# Binomial coefficients and generating functions

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## Binomial coefficients

The binomial coefficients $\binom{n}{k}$ counts the number of ways one can choose $k$ objects from given $n$. They are coefficients in the binomial expansion

$$(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}x + \binom{n}{n}.$$

More explicitly,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

A very useful formula for the binomial coefficients is

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Here are two simple exercises about binomial coefficients.

- Prove that if $n = 2^m$ with $m$ a positive integer, then
$$\binom{n}{k}$$
is an even integer for any $1 \le k \le n-1$.
- Let $m$ and $n$ be integers such that $1 \le m \le n$. Prove that $m$ divides the number
$$n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k}.$$

## Generating functions

The terms of a sequence $(a_n)_{n\ge 0}$ can b e combined into a function

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

called the generating function of the sequence. For example, the finite sequence $\binom{m}{n}$, with $m$ fixed and $n$ varies, gives the function $(x+1)^m$. The generating function for $a_n = \frac{1}{n}$ is $-\ln(1-x)$.

Generating functions provide a method to understand recursive relations of a sequence.

**Theorem.** *Suppose $a_n$ $(n \ge 0)$ is a sequence satisfying a second-order linear recurrence,*

$$a_n + ua_{n-1} + va_{n-2} = 0.$$

*Suppose that the quadratic equation $\lambda^2 + u\lambda + v = 0$ has two distinct roots $r_1, r_2$. Then*

$$a_n = \alpha r_1^n + \beta r_2^n$$

*for some real numbers $r_1, r_2$.*

*Proof.* Let $G(x) = a_0 + a_1x + a_2x^2 + \cdots$ be the generating function of $a_n$. The recurrence relation of $a_n$ implies that

$$G(x) - a_0 - a_1x + u\big(G(x) - a_0\big) + vx^2G(x) = 0.$$

Solving for $G(x)$, we have

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{1 + ux + vx^2} = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)}.$$

Using partial fractions, we have

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)} = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x} = \sum_{n=1}^{\infty} \big(\alpha r_1^n + \beta r_2^n\big)x^n.$$

Therefore, $a_n = \alpha r_1^n + \beta r_2^n$. ∎

1. Prove the identity

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

2. Give two proofs of the identity

$$\sum_{j=0}^{n} 2^{n-j}\binom{n}{j}\binom{j}{\lfloor j/2 \rfloor} = \binom{2n+1}{n}.$$

## Problems

1. Prove the identity

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

2. Find the general formula for the sequence $(y_n)_{n\ge 0}$ with $y_0 = 1$ and $y_n = ay_{n-1} + b^n$ for $n \ge 1$, where $a$ and $b$ are two fixed distinct real numbers.

3. Prove that the Fibonacci numbers $F_n$ satisfy

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

4. Consider the triangular $n \times n$ matrix

$$A = \left\{\begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}\right\}.$$

   Compute the matrix $A^k$ for $k \ge 1$.

5. Show that the coefficient of $x^k$ in the expansion of $(1 + x + x^2 + x^3)^n$ is

$$\sum_{j=0}^{k} \binom{n}{k}\binom{n}{k-2j}.$$

6. Find

$$\binom{n}{1}1^2 + \binom{n}{2}2^2 + \binom{n}{3}3^2 + \cdots + \binom{n}{n}n^2.$$

7. Let $S_0$ be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows. The integer $a$ is in $S_{n+1}$ if and only if exactly one of $a-1$ or $a$ is in $S_n$. Show that there are infinitely many integers $N$ for which

$$S_N = S_0 \cup \{N + a \mid a \in S_0\}.$$

8. For a set $S$ of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs $(s_1, s_2)$ such that

$$s_1, s_2 \in S, s1 \neq s2, \quad \text{and} \quad s1 + s2 = n.$$

Is it possible to partition the nonnegative integers into two sets $A$ and $B$ in such a way that $r_A(n) = r_B(n)$ for all $n$?

9. For positive integer $n$, denote by $S(n)$ the number of choices of the signs "+" or "−" such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Prove that

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos t \cos 2t \cdots \cos nt \, dt.$$

10. Let $S_n$ denote the set of all permutations of the numbers $1, 2, \ldots, n$. For$\pi \in S_n$, let $\sigma(\pi) = 1$ if $\pi$ is an even permutation and $\sigma(\pi) = -1$ if $\pi$ is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of $\pi$. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$