Polynomials, factors and the Viète relations

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Factors of a polynomial

Theorem. *Let $P(x_1, \dots, x_n), Q(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ be two polynomials in n variables. Here $\mathbb{R}$ can be replaced by any other field. Suppose $Q(x_1, \dots, x_n)$ is irreducible, and suppose $P(x_1, \dots, x_n) = 0$ whenever $Q(x_1, \dots, x_n) = 0$. Then $P(x_1, \dots, x_n)$ is divisible by $Q(x_1, \dots, x_n)$. In other words, $\frac{P(x_1,\dots,x_n)}{Q(x_1,\dots,x_n)}$ is a polynomial. Moreover, if both of P, Q are of integer coefficients, and if the gcd of the coefficients of Q is 1, then $\frac{P(x_1,\dots,x_n)}{Q(x_1,\dots,x_n)}$ has integer coefficients.*

Example. *Given a polynomial $P(x, y, z)$, prove that the polynomial*

$$Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)$$

is divisible by $(x - y)(y - z)(z - x)$.

Viète's relations

From the fundamental theorem of algebra, it follows that a polynomial with complex number coefficients

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \qquad (a_n \neq 0)$$

can be factored as

$$P(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n).$$

Equating the coefficients of x in the two expressions, we obtain

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= -\frac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n &= \frac{a_{n-2}}{a_n} \\ &\cdots \\ x_1x_2\cdots x_n &= (-1)^n \frac{a_0}{a_n}. \end{aligned}$$

Example. *If the quartic $x^4 + 3x^3 + 11x^2 + 9x + A$ has roots k, l, m, and n such that $kl = mn$, find A.*

More exercises about polynomials

The problems are not necessarily related to the above two methods.

1. Find all polynomials satisfying the functional equation

$$(x + 1)P(x) = (x - 6)P(x + 1).$$

2. Let a, b, c be real numbers. Show that $a \geq 0$, $b \geq 0$ and $c \geq 0$ if and only if $a + b + c \geq 0$, $ab + bc + ca \geq 0$ and $abc \geq 0$.

3. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros

$$x_1 < x_2 < \cdots < x_{n-1} < x_n$$

are real. Prove that

$$P'(\frac{x_1 + x_2}{2}) \cdot P'(\frac{x_{n-1} + x_n}{2}) \neq 0.$$

4. Let x_1 and x_2 be the roots of the equation $x^2 + 3x + 1 = 0$. Compute

$$\left(\frac{x_1}{x_2 + 1}\right)^2 + \left(\frac{x_2}{x_1 + 1}\right)^2.$$

5. If $x^2 + \frac{1}{x^2} = 14$ and $x > 0$, what is the value of $x^5 + \frac{1}{x^5}$?

6. In $x^3 + px^2 + qx + r = 0$, one zero is the sum of the other two. What is the relation between p, q and r?

7. Prove that if $P(x)$ is a polynomial with integer coefficients, and there exists a positive integer k such that none of the integers $P(1), P(2), ..., P(k)$ is divisible by k, then $P(x)$ has no integral root.

8. Suppose that the function $f(x) = ax^2 + bx + c$, where a, b, c are real constants, satisfies the condition $|f(x)| \leq 1$ for $|x| \leq 1$. Prove that $|f'(x)| \leq 4$ for $|x| \leq 1$.

9. Let $P(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1. Suppose one of its roots is equal to the product of the other two. Show that $2P(-1)$ is a multiple of $P(1) + P(-1) - 2(1 + P(0))$.

10. If $x + y + z = 0$, prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}.$$

11. Let $P(x)$ be a polynomial of degree n. Knowing that

$$P(k) = \frac{k}{k+1}, \quad k = 0, 1, \ldots, n,$$

find $P(M)$ for $m > n$.

12. Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a + 1)^n = 2001.$$